9/11/25 Lecture 6

Conjecture: A statement without a proof (essentially a guess)

Intro to Proofs (2.2)

- Often the universal quantifier is implied
- Proof by exhaustion = show it one by one
- Proof by universal generalization (MAYBE DELETE THIS)

Example x, y are positive integers

Prove $\frac{x}{y} + \frac{y}{x} \geq 2$

"$\forall x \forall y [x > 0 \wedge y > 0 \rightarrow \frac{x}{y} + \frac{y}{x} \geq 2]$"

or "$\forall x \in \mathbb{Z}^+ \ \forall y \in \mathbb{Z}^+ \ \frac{x}{y} + \frac{y}{x} \geq 2$"

Going back ward

$\Leftrightarrow \frac{x^2+y^2}{xy} \geq 2$

$\Leftrightarrow x^2+y^2 \geq 2xy$

$\Leftrightarrow x^2+y^2-2xy \geq 0$

$\Leftrightarrow (x-y)^2 \geq 0$ this is always true for positive integers so $\frac{x}{y} + \frac{y}{x} \geq 2$ is also always true.

Notation:

$4 \mid x$ "X can be divided by 4" or "4 divides x"

$4 \nmid x$ "4 does NOT divide x"

$\boxed{\text{Example}}$ x is positive integer
$x \le x^2$

"$\forall x \in \mathbb{Z}^+ \quad x \le x^2$"

$$x \le x^2$$
$$\equiv x - x^2 \le 0$$
$$\equiv x(1-x) \le 0$$

Now we have 2 cases!

when $x = 1$ True

when $x \ge 2$ this is x times negative something so the result is always negative.

n is an odd integer

$n = 2k+1$ for some $k \in \mathbb{Z}$ then $4 \mid (n^2-1)$ "(n^2-1) is divisble by 4"

Sum of squares of two consecutive integers is odd

$$\begin{array}{r} (x)^2 \\ + (x+1)^2 \\ \hline \text{odd} \end{array}$$

$$x^2 + x^2 + 2x + 1$$
$$= 2x^2 + 2x + 1$$
$$= 2(x^2+x) + 1 = \text{odd because } 2(x^2+x) \text{ is even}$$

Proof by contrapositive (2.5)

Example

$\forall x \forall y \quad xy > 400 \rightarrow x > 20 \lor y > 20$ for positive real numbers

Contrapositive $\equiv \neg(x > 20 \lor y > 20) \rightarrow \neg(xy > 400)$

$\equiv x \leq 20 \land y \leq 20 \rightarrow xy \leq 400$ DeMorgan Law

$x \leq 20$

$\equiv xy \leq 20y$

$y \leq 20 \Rightarrow 20y \leq 400$

Example

$$\left.\begin{array}{l} x \mid (y+1) \\ x \geq 2 \end{array}\right\} \quad x \nmid y$$

for proof by contrapositive assume that $x \mid y$

$\exists a \in \mathbb{Z} \quad y+1 = ax \qquad 1 = ax - bx$

$\exists b \in \mathbb{Z} \quad y = bx \qquad 1 = x(a-b)$

$\Rightarrow a > b \qquad \left[\frac{1}{a-b} = x\right] \equiv F$

$$\forall x \in \mathbb{Z}^+ \quad \forall y \in \mathbb{Z}^+ \quad (x > 1 \land (x \mid y)) \rightarrow x \nmid (y+1)$$

$$\equiv \forall x > 1 \quad \forall y \left[x \mid y \rightarrow x \nmid (y+1) \right]$$

$$\equiv \forall x > 1 \quad \forall y \in \underbrace{\{ y \mid x \text{ divides } y \}}_{\substack{\text{all } y \text{ that} \\ \text{can be divided by } x}} \quad x \nmid (y+1)$$